

The art of PD curve calibration

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PD curve calibration refers to the task of transforming a set of conditional probabilities of default (PDs) to another average PD level that is determined by a change of the underlying unconditional PD. This paper presents a framework that allows to explore a variety of calibration techniques and the conditions under which they are fit for purpose. We test the techniques discussed by applying them to a publicly available dataset of agency rating and default statistics that can be considered typical for the scope of application of the techniques. We show that the popular technique of ‘scaling the PD curve’ is theoretically questionable and does not perform well on the test datasets. We identify two calibration techniques that are both theoretically sound and perform much better on the test datasets.

KEYWORDS: Probability of default, calibration, likelihood ratio, Bayes’ formula, rating profile.

1. Introduction

The best way to understand the subject of this paper is to have a glance at table 1 that illustrates its main results. Table 1 compares the default rates (column 2) that were observed in 2010 and 2011 for non-financial corporates with different S&P rating grades (S&P, 2011, 2012) to default rate forecasts that were made with four different techniques discussed in this paper (columns 3, 4, 5 and 6). The process of forecasting default rates for the grades of a rating system is called *calibration*, and the resulting forecast default rates are called *PD (probability of default) curve*. Although some forecast techniques (columns 4 and 6) show a better performance in table 1 than the others, there is no technique that is better than the others in both forecast years. All forecast techniques ‘fail’ in predicting the rating grades with 0% default rate – but as explained below this is intended. Table 12 in section 4 (for a different universe of S&P rated names) demonstrates another deliberate ‘failure’ of the forecast techniques: While the observed default rates are non-

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monotonic all the forecast default rates are increasing with deteriorating credit quality. This again is an intended effect.

The subject of this paper basically is to explain

- how the forecasts in tables 1 and 12 were made, and
- how the performance of the forecast approaches can be measured and compared.

The scope of the concepts and techniques described in this paper is not limited to data from rating agencies but covers any rating system for which data from an *estimation period* is available. It should, however, be noted that the focus in this paper is on grade-level default rate forecasts while the problem of forecasting the unconditional (or portfolio-wide) default rate is not considered. Forecasting the unconditional default rate is an econometric problem that is beyond the scope of this paper (see Engelmänn and Porath, 2012, for an example of how to approach this problem).

This paper appears to be almost unique in that it solely deals with the calibration or recalibration of PD curves. Calibration of PD curves is a topic that is often mentioned in the literature but mostly only as one aspect of the more general subject of rating model development. For instance, Falkenstein et al. (2000) deployed the approach that is called ‘invariant PDs’ in table 1 without any comment about why they considered it appropriate. There are, however, some authors who devoted complete articles or book sections to PD curve calibration. Van der Burgt (2008) suggested a predecessor of the technique that is called quasi moment matching (QMM) in this paper. Bohn and Stein (2009, chapter 4) discussed the conceptual and practical differences between the ‘scaled PDs’ and ‘invariant likelihood’ techniques. More recently, Konrad (2011) investigated in some detail the interplay between the calibration and the discriminatory power of rating models.

In this paper, we revisit the concept of two calibration steps as used by Bohn and Stein (2009). According to Bohn and Stein (2009) the two steps are a consequence of the fact that usually the first calibration of a rating model is conducted on a training sample in which the proportion of good and bad might not be representative of the live portfolio. The second calibration step, therefore, is needed to adjust the calibration to the right proportion of good and bad.

We argue more generally that the two steps actually relate to different time periods (the estimation and the forecast periods) which both can be described by the same type of one-period model. This view encompasses both the situation where a new rating model is calibrated and the situation where an existing rating model undergoes a – possibly periodic – recalibration. The estimation period is used to estimate the model components that are assumed to be invariant (i.e. unchanged) or in a specific way transformed between the estimation and the forecast periods. Calibration techniques for the forecast period are essentially determined by the assumptions of invariance between the periods.

The roadmap for this paper is as follows:

- In the following section 2, we present a dataset that is typical as a starting point for the sort of calibration we are going to consider in the remainder of the paper. We also discuss the constraints to which the calibration approach must be subject and explore their appropriateness with statistical tests.

Table 1: 2010 and 2011 grade-level forecast default rates for S&P non-financial corporates ratings. P-values are for the χ^2 -tests of the implied default profiles. All values in %.

	Default rate	Invariant default profile (4.10)	Invariant AR (4.11)	Scaled PDs (4.12)	Scaled likelihood ratio (4.13)
	2010: Unconditional default rate 1.423				
AAA	0	0.0005	0.0001	0.0002	0.0001
AA+	0	0.001	0.0002	0.0005	0.0003
AA	0	0.0022	0.0005	0.0011	0.0007
AA-	0	0.0042	0.0011	0.0021	0.0014
A+	0	0.0069	0.002	0.0037	0.0025
A	0	0.0131	0.0045	0.0075	0.005
A-	0	0.0262	0.0107	0.0163	0.0108
BBB+	0	0.0476	0.0223	0.0317	0.0211
BBB	0	0.0836	0.046	0.0598	0.0397
BBB-	0	0.1391	0.0881	0.1062	0.0707
BB+	0	0.2011	0.1355	0.1614	0.1076
BB	0	0.2758	0.196	0.2314	0.1546
BB-	0	0.4249	0.3175	0.3797	0.255
B+	0	0.7616	0.6005	0.7402	0.5036
B	0.7634	1.6652	1.4378	1.7877	1.2639
B-	2.0747	4.4114	4.1636	5.1613	4.1758
CCC-C	21.9895	14.1473	15.7635	13.2909	16.4903
P-value	Exact	12.02	23.46	9.5	29.77
	2011: Unconditional default rate 0.945				
AAA	0	0.0002	0.0001	0.0002	0.0001
AA+	0	0.0004	0.0002	0.0004	0.0003
AA	0	0.0009	0.0004	0.0009	0.0006
AA-	0	0.0018	0.0009	0.0017	0.0012
A+	0	0.0031	0.0016	0.003	0.0021
A	0	0.0063	0.0035	0.0061	0.0043
A-	0	0.0134	0.008	0.0132	0.0093
BBB+	0	0.0258	0.016	0.0257	0.0182
BBB	0	0.0477	0.0308	0.0483	0.0343
BBB-	0	0.0833	0.0572	0.0859	0.0612
BB+	0	0.1245	0.0885	0.1305	0.0931
BB	0	0.1758	0.1252	0.1871	0.1338
BB-	0	0.2817	0.1979	0.307	0.2208
B+	0.2237	0.5327	0.3824	0.5985	0.4361
B	0.7782	1.2522	1.0389	1.4455	1.0957
B-	4.2636	3.6293	3.701	4.1734	3.6343
CCC-C	16.8067	13.1196	15.0199	10.7469	14.5952
P-value	Exact	69.43	91.6	38.71	86.98

- Section 3 provides the theoretical framework that is needed for dealing with the technical aspects of PD curve calibration. In particular, we derive a result (proposition 3.2) on the characterisation of the one-period model by unconditional rating profile and likelihood ratio. This result proves useful for the identification of proper and accurate calibration techniques.
- In section 4 we describe a variety of techniques for PD curve calibration and examine their performance with a real data example (the S&P data presented in section 2). The suitability of the techniques described depends strongly upon what data (e.g. the unconditional rating profile) can be observed at the time when the forecast exercise takes place. We therefore discuss the different possibilities and assumptions in some detail. It turns out that the popular ‘scaling the PD curve’ approach is both theoretically questionable and not very well performing on the example dataset. We identify two techniques (quasi moment matching and ‘scaling the likelihood ratio’) that are theoretically sound and perform much better when deployed for the numerical example.
- Section 5 concludes the paper.

2. Data and context

All numerical examples in this paper are based on the S&P rating and default statistics for all corporates and non-financial corporates only as presented in table 2. Only with their 2009 default data report [S&P \(2010\)](#) began to make information on modified-grade level issuer numbers readily available. Without issuer numbers, however, there is not sufficient information to calculate rating profiles and conduct goodness-of-fit tests for rating profiles. This explains why we only look at default statistics from 2009 onwards. For the purposes of this paper, data from Moody’s is less suitable because Moody’s do not provide issuer numbers at alphanumeric grade level and apply a special treatment to withdrawn ratings that creates undesirable uncertainty with regard to samples sizes ([Moody’s, 2012](#)).

We chose to look at S&P’s non-financial corporates default statistics (lower panel of table 2) as an example of a ‘nice’ dataset where the observed default rates increase with deteriorating credit quality. S&P’s all corporates default statistics (upper panel of table 2) represent an example of a more interesting, somewhat problematic dataset because it includes some instances of *inversions* of observed default rates. ‘Inversion of default rates’ means that the default rate observed for a better rating grade is higher than the default rate of the adjacent worse rating grade.

2.1. Default rate inversions

Table 3 shows that for non-financial corporates the grade-level default rates recorded by S&P for 2009, 2010 and 2011 increase with deteriorating credit quality as one would expect. In contrast, there are a number of ‘inversions’ in the all corporates columns of the table, i.e. there are some counter-intuitive examples of adjacent rating grades where the less risky grade has a higher default rate than the adjacent riskier grade. Notable for this phenomenon is, in particular, the pair of BBB- and BB+ in 2009 with 1.09% defaults in BBB- and 0% defaults in BB+.

Table 2: S&P's corporate ratings and defaults in 2009, 2010 and 2011. Sources: [S&P \(2010, tables 51 to 53\)](#), [S&P \(2011, tables 50 to 52\)](#), [S&P \(2012, tables 50 to 52\)](#).

	2009		2010		2011	
Rating grade	# rated	# defaults	# rated	# defaults	# rated	# defaults
	All corporates					
AAA	81	0	72	0	51	0
AA+	37	0	25	0	36	0
AA	188	0	143	0	120	0
AA-	245	0	209	0	207	0
A+	340	1	353	0	357	0
A	510	2	474	0	470	0
A-	546	0	528	0	560	0
BBB+	498	2	457	0	473	0
BBB	541	1	583	0	549	0
BBB-	459	5	430	0	508	1
BB+	266	0	254	2	260	0
BB	295	3	276	1	319	0
BB-	441	4	379	2	403	0
B+	438	24	393	0	509	2
B	482	48	436	3	586	7
B-	303	52	290	6	301	12
CCC-C	190	92	220	49	138	22
All	5860	234	5522	63	5847	44
	Non-financial corporates					
AAA	15	0	12	0	13	0
AA+	13	0	8	0	6	0
AA	57	0	49	0	41	0
AA-	62	0	50	0	46	0
A+	91	0	84	0	85	0
A	215	0	197	0	205	0
A-	283	0	275	0	274	0
BBB+	301	0	277	0	301	0
BBB	370	0	392	0	374	0
BBB-	310	0	287	0	357	0
BB+	205	0	184	0	193	0
BB	240	2	222	0	249	0
BB-	355	4	296	0	327	0
B+	389	20	341	0	447	1
B	436	43	393	3	514	4
B-	259	50	241	5	258	11
CCC-C	167	86	191	42	119	20
All	3768	205	3499	50	3809	36

Table 3: S&P grade-level default rates for corporates in 2009, 2010 and 2011. Sources: [S&P \(2010, tables 51 to 53\)](#), [S&P \(2011, tables 50 to 52\)](#), [S&P \(2012, tables 50 to 52\)](#). All values in %.

	All corporates			Non-financial corporates		
Rating grade	2009	2010	2011	2009	2010	2011
AAA	0.00	0.00	0.00	0.00	0.00	0.00
AA+	0.00	0.00	0.00	0.00	0.00	0.00
AA	0.00	0.00	0.00	0.00	0.00	0.00
AA-	0.00	0.00	0.00	0.00	0.00	0.00
A+	0.29	0.00	0.00	0.00	0.00	0.00
A	0.39	0.00	0.00	0.00	0.00	0.00
A-	0.00	0.00	0.00	0.00	0.00	0.00
BBB+	0.40	0.00	0.00	0.00	0.00	0.00
BBB	0.18	0.00	0.00	0.00	0.00	0.00
BBB-	1.09	0.00	0.20	0.00	0.00	0.00
BB+	0.00	0.79	0.00	0.00	0.00	0.00
BB	1.02	0.36	0.00	0.83	0.00	0.00
BB-	0.91	0.53	0.00	1.13	0.00	0.00
B+	5.48	0.00	0.39	5.14	0.00	0.22
B	9.96	0.69	1.19	9.86	0.76	0.78
B-	17.16	2.07	3.99	19.31	2.07	4.26
CCC-C	48.42	22.27	15.94	51.50	21.99	16.81
All	3.99	1.14	0.75	5.44	1.43	0.95

Should we conclude from the existence of such inversions that there is a problem with the rank-ordering capacity of the rating methodology? The long-run average grade-level default rates reported by [S&P \(2012, Table 23\)](#) suggest that the observation of inversions as in Table 3 might be an exception. Fisher's exact test ([Fisher, 1922](#); [Casella and Berger, 2002](#), Example 8.3.30) allows to verify this explanation.

The test can be described as follows:

- Let Y_i , $i = 1, 2$ be independent and binomially distributed random variables with sizes n_i , $i = 1, 2$ and success probabilities p_i , $i = 1, 2$.
- The Null hypothesis is $p_1 \leq p_2$, against the alternative $p_1 > p_2$.
- Under the Null hypothesis the distribution of Y_1 conditional on $Y_1 + Y_2$ is hypergeometric:

$$P[Y_1 = y_1 | Y_1 + Y_2 = y] = \frac{\binom{n_1}{y_1} \binom{n_2}{y-y_1}}{\binom{n_1+n_2}{y}}. \quad (2.1a)$$

- Assume that y_1 successes were observed for Y_1 and y successes for Y_1 and Y_2 together.

Table 4: p-values for Fisher’s exact test of ‘2009 all corporates PD of better grade was less than or equal to PD of worse grade’ against ‘PD of better grade was greater than PD of worse grade’. Data are the rating and default numbers from columns 2 and 3 in table 2.

Null hypothesis	Alternative	p-value
$PD(AAA) \leq PD(AA+)$	$PD(AAA) > PD(AA+)$	1
$PD(AA+) \leq PD(AA)$	$PD(AA+) > PD(AA)$	1
$PD(AA) \leq PD(AA-)$	$PD(AA) > PD(AA-)$	1
$PD(AA-) \leq PD(A+)$	$PD(AA-) > PD(A+)$	1
$PD(A+) \leq PD(A)$	$PD(A+) > PD(A)$	0.785
$PD(A) \leq PD(A-)$	$PD(A) > PD(A-)$	0.233
$PD(A-) \leq PD(BBB+)$	$PD(A-) > PD(BBB+)$	1
$PD(BBB+) \leq PD(BBB)$	$PD(BBB+) > PD(BBB)$	0.469
$PD(BBB) \leq PD(BBB-)$	$PD(BBB) > PD(BBB-)$	0.991
$PD(BBB-) \leq PD(BB+)$	$PD(BBB-) > PD(BB+)$	0.101
$PD(BB+) \leq PD(BB)$	$PD(BB+) > PD(BB)$	1
$PD(BB) \leq PD(BB-)$	$PD(BB) > PD(BB-)$	0.582
$PD(BB-) \leq PD(B+)$	$PD(BB-) > PD(B+)$	1
$PD(B+) \leq PD(B)$	$PD(B+) > PD(B)$	0.996
$PD(B) \leq PD(B-)$	$PD(B) > PD(B-)$	0.999
$PD(B-) \leq PD(CCC-C)$	$PD(B-) > PD(CCC-C)$	1

Reject then the Null hypothesis at level α if $\alpha \geq \text{p-value}$, where

$$\begin{aligned}
 \text{p-value} &= P[Y_1 \geq y_1 \mid Y_1 + Y_2 = y] \\
 &= \sum_{k=y_1}^{\min(n_1, y)} \frac{\binom{n_1}{k} \binom{n_2}{y-k}}{\binom{n_1+n_2}{y}}.
 \end{aligned} \tag{2.1b}$$

In order to apply Fisher’s test to the S&P data we revisit table 2. The results of the 16 tests applied are presented in table 4. For instance, to obtain the p-value of 10.1% for the test of ‘PD of BBB- is less than or equal to PD of BB+’ against the alternative ‘PD of BBB- is greater than PD of BB+’ we apply formula (2.1b) with $n_1 = 459$, $n_2 = 266$, $y_1 = 5$ and $y = 5$. As a matter of fact, 10.1% is the least p-value reported in table 4. Hence, we cannot conclude that there are statistically significant inversions in the grade-level all corporates default rates although the inversion involving BBB- and BB+ is nearly significant at 10% type-I error level. But even if this inversion had been significant we should not have been surprised because when 16 tests are conducted there is a high probability that at least one of them will erroneously indicate rejection of the Null hypothesis.

As a conclusion from the discussion of default rate conversions we will assume for the remainder of the paper that the PD curve associated with the corporate S&P ratings can be modelled as a monotonic curve.

Table 5: S&P rating profiles for corporates at the beginning of 2009, 2010 and 2011. Sources: [S&P \(2010, tables 51 to 53\)](#), [S&P \(2011, tables 50 to 52\)](#), [S&P \(2012, tables 50 to 52\)](#) and own calculations. All values in %.

	All corporates			Non-financial corporates		
Rating grade	2009	2010	2011	2009	2010	2011
AAA	1.38	1.30	0.87	0.40	0.34	0.34
AA+	0.63	0.45	0.62	0.35	0.23	0.16
AA	3.21	2.59	2.05	1.51	1.40	1.08
AA-	4.18	3.78	3.54	1.65	1.43	1.21
A+	5.80	6.39	6.11	2.42	2.40	2.23
A	8.70	8.58	8.04	5.71	5.63	5.38
A-	9.32	9.56	9.58	7.51	7.86	7.19
BBB+	8.50	8.28	8.09	7.99	7.92	7.90
BBB	9.23	10.56	9.39	9.82	11.20	9.82
BBB-	7.83	7.79	8.69	8.23	8.20	9.37
BB+	4.54	4.60	4.45	5.44	5.26	5.07
BB	5.03	5.00	5.46	6.37	6.34	6.54
BB-	7.53	6.86	6.89	9.42	8.46	8.58
B+	7.47	7.12	8.71	10.32	9.75	11.74
B	8.23	7.90	10.02	11.57	11.23	13.49
B-	5.17	5.25	5.15	6.87	6.89	6.77
CCC-C	3.24	3.98	2.36	4.43	5.46	3.12
All	100.00	100.00	100.00	100.00	100.00	100.00

2.2. Pearson's chi-squared test for count data

A question of similar importance for the estimation of PD curves is the question of whether or not the unconditional (or all-portfolio) rating profile (i.e. the distribution of the rating grades) of a portfolio can be assumed to be unchanged over time. Table 5 shows the unconditional rating profiles of all corporates and the non-financial corporates only for the three years for which there are publicly available S&P data. It is hard to assess from the percentages whether the profiles are broadly invariant during the three years and whether or not the profiles of all corporates and the non-financial corporates only are identical up to random differences. We use Pearson's χ -squared test for count data ([Pearson, 1900](#)) to answer these questions and also to assess the accuracy of the forecast techniques discussed in the remainder of the paper.

Description of the χ -squared test. Assume that on a set $\{1, \dots, k\}$ of $k > 1$ classes we have made a total of $n > 0$ independent observations which each falls into one of the classes. Hence

Table 6: χ -squared tests for identity of rating profiles from table 5. All p-values have been calculated by χ -squared approximation.

Class probabilities	Observed frequencies	p-value
Corporate profile 2009	Corporates 2010	0.000616
Corporate profile 2009	Corporates 2011	$< 10^{-10}$
Corporate profile 2010	Corporates 2011	$< 10^{-10}$
Non-financial profile 2009	Non-financials 2010	0.09314
Non-financial profile 2009	Non-financials 2011	6.3×10^{-7}
Non-financial profile 2010	Non-financials 2011	$< 10^{-10}$
Corporate profile 2009	Non-financials 2009	$< 10^{-10}$
Corporate profile 2010	Non-financials 2010	$< 10^{-10}$
Corporate profile 2011	Non-financials 2011	$< 10^{-10}$

there are n_i observations with class i and it holds that

$$n = \sum_{i=1}^k n_i. \quad (2.2a)$$

We want to test the hypothesis that the observations (n_1, \dots, n_k) are the realisation of a multinomial distribution with class probabilities $q_1 > 0, \dots, q_{k-1} > 0, q_k = 1 - \sum_{i=1}^{k-1} q_i > 0$. Pearson suggested the test statistic

$$T = \sum_{i=1}^k \frac{(n_i - n q_i)^2}{n q_i}. \quad (2.2b)$$

It is well known that for large n the statistic T is approximately χ -squared distributed with $k - 1$ degrees of freedom (see, e.g., [van der Vaart, 1998](#), Theorem 17.2). As to the question of when the approximation may be applied, it is common practice to use it when all expected class frequencies under the hypothesis are greater than or equal to five, i.e.

$$n q_i \geq 5 \quad \text{for all } i = 1, \dots, k. \quad (2.2c)$$

If this condition is violated we can still approximate the distribution of T by Monte-Carlo simulation. For the purpose of this paper, we ran the Monte-Carlo simulation with 5,000 iterations when we could not apply the χ -squared approximation.

Table 6 shows that

- the all corporates rating profiles change significantly over the years,
- in all three years the all corporates and the non-financial corporates rating profiles are significantly different,
- also the non-financial corporates rating profiles change a lot over the years. Only the 2009 and 2010 profiles can be seen as statistically indistinguishable at 5% type-I error level.

2.3. Consequences for the calibration of PD curves

From the initial data analyses in sections 2.1 and 2.2 we can draw two conclusions:

- Forcing monotonicity of estimated PD curves can make sense if it is justified by statistical tests or long-run average evidence.
- In general, we cannot assume that the rating profile of a portfolio does not change over time, even if random fluctuation is ignored. However, this assumption can be verified or proven wrong with statistical tests. As we will see in the section 4, depending on the outcome of the tests there are different options for the forecast of PD curves.

Although never a default of an AAA-rated corporate was observed within one year after having been rated AAA (Moody's, 2012; S&P, 2012) we should nonetheless try and infer a positive one-year PD for AAA. For the possibility of default technically is not excluded even for AAA rated (or any other investment grade) entities. In particular, even if we may assume that the rating agencies will not allow AAA rated companies to default without having down-graded them in time it might happen that an AAA company's management for whatever reason decides to have the company defaulting.

This is why in the following we restrict ourselves to only deploy PD curve estimation techniques that guarantee to deliver positive PDs for all rating grades.

On the basis of the data presented in this section, it is also worthwhile to clarify precisely the concept of a two-step (or two-periods) approach to the calibration of a rating model as mentioned by Bohn and Stein (2009): The first period is the *estimation* period, the second period is the *calibration and forecast* period. The two periods are determined by their start and end dates and the observation and estimation horizon:

- h is the horizon for the PD estimation, i.e. a borrower's PD at date T gives the probability that the borrower will default within T and $T + h$.
- The start date T_0 of the estimation period is a date in the past.
- $T_1 \geq T_0 + h$ is the date when the calibration or recalibration of the rating model takes place. The calibration is for the current portfolio of borrowers whose ratings at T_1 should be known but whose future default status at $T_1 + h$ is still unknown.
- The end date $T_2 = T_1 + h$ of the forecast period is in the future. Then the default status of the borrowers in the current portfolio will be known.

With regard to the two-periods concept for calibration, for the remainder of the paper we make the following crucial assumptions:

- For the sample as of date T_0 everything is known:
 - The unconditional rating profile at T_0 ,
 - the conditional rating profiles (i.e. conditional on default and conditional on survival respectively) at T_0 ,
 - the unconditional (base) PD for the time interval between T_0 and $T_0 + h$,
 - the conditional PDs (i.e. conditional on the rating grades at T_0 for the time interval

between T_0 and $T_0 + h$.

- At date T_1 could be known:
 - The unconditional rating profile.
 - A forecast of the unconditional (base) default rate. In general this will be different from the unconditional PD for the estimation period between T_0 and $T_0 + h$.

We will use the rating and default data for 2009 from table 2 as an example for the estimation period (i.e. $h = 1$ year, $T_0 = \text{January 1, 2009}$, $T_0 + h = \text{December 31, 2009}$). We will consider both 2010 and 2011 as examples of one-year forecast periods based on the estimation of a model for 2009 (i.e. $T_1 = \text{January 1, 2010}$ or $T_1 = \text{January 1, 2011}$). The gap of one year between the estimation period 2009 and the forecast period 2011 reflects the gap that is likely to occur in practice when the full data from the estimation period usually become available only a couple of months after the end of the period.

3. One-period mechanics

This section describes a one-period statistical model of a borrower's beginning of the period rating grade and end of the period state of solvency. This model is applicable to both the estimation and the forecast periods as discussed in section 2.3. In particular, we will consider the following model characteristics and their relationships:

- Unconditional rating distribution (profile).
- Conditional (on default and survival) rating distributions (profiles).
- Unconditional PD.
- PD curve (PDs conditional on rating grades).
- Accuracy ratio (discriminatory power).
- Likelihood ratio.

We rely on the standard model used for topics like pattern recognition, medical diagnoses, or signal detection (see, e.g., [van Trees, 1968](#)). [Hand \(1997\)](#) presents a variety of applications (including credit scoring) for this type of model.

Speaking in technical terms, in this paper we study the joint distribution and some estimation aspects of a pair (X, S) of random variables. The variable X is interpreted as the *rating grade*¹ assigned to a solvent borrower at the beginning of the observation period. Hence X typically takes on values on a discrete scale in a finite set which we describe without loss of generality as $\{1, 2, \dots, k\}$. This implies that the marginal distribution of X is characterised by the probabilities $\Pr[X = x]$, $x = 1, \dots, k$, which we call the *unconditional rating profile*.

¹In practice, often a rating model with a small finite number of grades is derived from a score function with values on a continuous scale. This is usually done by mapping score intervals on rating grades. See [Tasche \(2008, section 3\)](#) for a discussion of how such mappings can be defined. Discrete rating models are preferred by practitioners because manual adjustment of results (overrides) is feasible. Moreover, results by discrete rating models tend to be more stable over time.

Convention: Low values of X indicate low creditworthiness (“bad”), high values of X indicate high creditworthiness (“good”).

The variable S is the *borrower’s state of solvency* at the end of the observation period, typically one year after the rating grade was observed. S takes on values in $\{0, 1\}$. The meaning of $S = 0$ is “borrower has remained solvent” (solvency or survival), $S = 1$ means “borrower has become insolvent” (default). In particular, S is always observed with a time lag to the observation of X . Hence, when S is observed X is already known but when X is observed today S is still unknown. We write D for the event $\{S = 1\}$ and N for the event $\{S = 0\}$. Hence

$$D \cap N = \{S = 1\} \cap \{S = 0\} = \emptyset, \quad D \cup N = \text{whole space.} \quad (3.1)$$

The marginal distribution of the state variable S is characterised by the *unconditional probability of default* p which is defined as

$$p = \Pr[D] = \Pr[S = 1] \in [0, 1]. \quad (3.2)$$

p is sometimes also called *base probability of default*. In the following we assume $0 < p < 1$ as the cases $p = 0$ and $p = 1$ are not of practical relevance.

3.1. Model specification

Recall that the two marginal distributions of X and S respectively do not uniquely determine the joint distribution of X and S . For easy reference we state in the following proposition the three equivalent standard ways to characterise the joint distribution.

Proposition 3.1 *The joint distribution of the pair (X, S) of the rating variable X and the state of the borrower variable S is fully specified in either of the following three ways:*

(i) *By the joint probabilities*

$$\begin{aligned} \Pr[X = x, S = 0] &= \Pr[\{X = x\} \cap N], \quad x = 1, \dots, k, \quad \text{and} \\ \Pr[X = x, S = 1] &= \Pr[\{X = x\} \cap D], \quad x = 1, \dots, k. \end{aligned} \quad (3.3a)$$

(ii) *By the unconditional PD $p = \Pr[D] = 1 - \Pr[N]$ and the distributions of X conditional on D and N respectively:*

$$\begin{aligned} \Pr[X = x | D] &= \frac{\Pr[\{X = x\} \cap D]}{p}, \quad x = 1, \dots, k, \quad \text{and} \\ \Pr[X = x | N] &= \frac{\Pr[\{X = x\} \cap N]}{1 - p}, \quad x = 1, \dots, k. \end{aligned} \quad (3.3b)$$

$x \mapsto \Pr[X = x | D]$ and $x \mapsto \Pr[X = x | N]$ are called the *conditional rating profiles* (conditional on default and survival respectively).

(iii) *By the unconditional rating profile $x \mapsto \Pr[X = x]$ and the conditional PDs*

$$\Pr[D | X = x] = \frac{\Pr[\{X = x\} \cap D]}{\Pr[X = x]}, \quad x = 1, \dots, k. \quad (3.3c)$$

$x \mapsto \Pr[D | X = x]$ is called the *PD curve associated with the grades $x = 1, \dots, k$.*

For further reference we note how the specification of the joint distribution of (X, S) given in proposition 3.1 (ii) implies the representation provided in proposition 3.1 (iii):

- By the law of total probability, the unconditional rating profile $\Pr[X = x]$, $x = 1, \dots, k$ can be calculated as

$$\Pr[X = x] = p \Pr[X = x | D] + (1 - p) \Pr[X = x | N]. \quad (3.4a)$$

- Bayes' formula implies the following representation of the PD curve $\Pr[D | X = x]$:

$$\Pr[D | X = x] = \frac{p \Pr[X = x | D]}{p \Pr[X = x | D] + (1 - p) \Pr[X = x | N]}. \quad (3.4b)$$

Also for further reference, we observe how the specification of the joint distribution of (X, S) given in proposition 3.1 (iii) implies the representation provided in proposition 3.1 (ii):

- Again by the law of total probability, the unconditional PD p can be calculated as

$$p = \sum_{x=1}^k \Pr[D | X = x] \Pr[X = x]. \quad (3.5a)$$

- With regard to the conditional rating profiles, it follows directly from the definition of conditional probability that

$$\Pr[X = x | D] = \Pr[D | X = x] \Pr[X = x] / p, \quad \text{and} \quad (3.5b)$$

$$\Pr[X = x | N] = (1 - \Pr[D | X = x]) \Pr[X = x] / (1 - p). \quad (3.5c)$$

The equivalence between equations (3.4a) and (3.4b) on the one hand and equations (3.5a) and (3.5b), (3.5c) on the other hand allows the calculation of one set of characteristics once the other set of characteristics is known. But the equivalence also represents a consistency condition that must be kept in mind if one of the characteristics is changed. In particular, if for a given unconditional rating profile there are independent estimates of the unconditional PD and the PD curve equation, (3.5a) becomes a crucial consistency condition.

In this paper, we use quasi moment matching (QMM) as described in appendix B to transform the grade-level empirical default rates into smoothed PD curves. As mentioned in section 2.3, such smoothing of the empirical PD curve is needed in order to

- force monotonicity of the PD curve and
- force the PDs to be positive.

Matching in this context means fitting a two-parameter curve to the empirically observed unconditional default rate and discriminatory power. The discriminatory power is measured as accuracy ratio whose general formula is given in (B.4a). Using the conditional rating profiles defined by (3.3b) the accuracy ratio can also be described by

$$\text{AR} = \sum_{x=2}^k \Pr[X = x | N] \Pr[X \leq x - 1 | D] - \sum_{x=1}^{k-1} \Pr[X = x | N] \Pr[X \geq x + 1 | D]. \quad (3.6)$$

3.2. Likelihood ratio

We will see in section 4 that the specification of the one-period model by unconditional rating profile and PD curve (see proposition 3.1 (iii)) is unsatisfactory if we want to combine a forecast period profile with an estimation period PD curve. For according to equations (3.4a) and (3.4b) both components depend upon the unconditional PD – which might be different in the estimation and forecast periods. The likelihood ratio is a concept closely related to the PD but avoids the issue of dependence on the unconditional PD.

In the context of credit ratings, it can be reasonably assumed that all components of the conditional rating profiles $\Pr[X = x | D]$ and $\Pr[X = x | N]$, $x = 1, \dots, k$ are positive. For otherwise, there would be rating grades with sure predictions of default and survival – which is unlikely to happen with real-world rating models. We can therefore define the *likelihood ratio* λ associated with the rating model:

$$\lambda(x) = \frac{\Pr[X = x | N]}{\Pr[X = x | D]}, \quad x = 1, \dots, k. \quad (3.7)$$

The likelihood ratio $\lambda(x)$ specifies how much more (or less) likely it is for a survivors's rating grade to come out as x than for a defaulter's rating grade. Observe that (3.4b) can be rewritten as

$$\Pr[D | X = x] = \frac{p}{p + (1 - p) \lambda(x)}, \quad x = 1, \dots, k. \quad (3.8a)$$

This is equivalent to an alternative representation of the likelihood ratio:

$$\lambda(x) = \frac{1 - \Pr[D | X = x]}{\Pr[D | X = x]} \frac{p}{1 - p}, \quad x = 1, \dots, k. \quad (3.8b)$$

By (3.8b), the likelihood ratio can alternatively be described as the ratio of the grade x odds of survival and the unconditional odds of survival. By (3.5b), (3.8a) also implies

$$\Pr[X = x | D] = \frac{\Pr[X = x]}{p + (1 - p) \lambda(x)}, \quad x = 1, \dots, k, \quad (3.9a)$$

and, by taking the sum of all $\Pr[X = x | D]$

$$1 = \sum_{x=1}^k \frac{\Pr[X = x]}{p + (1 - p) \lambda(x)}. \quad (3.9b)$$

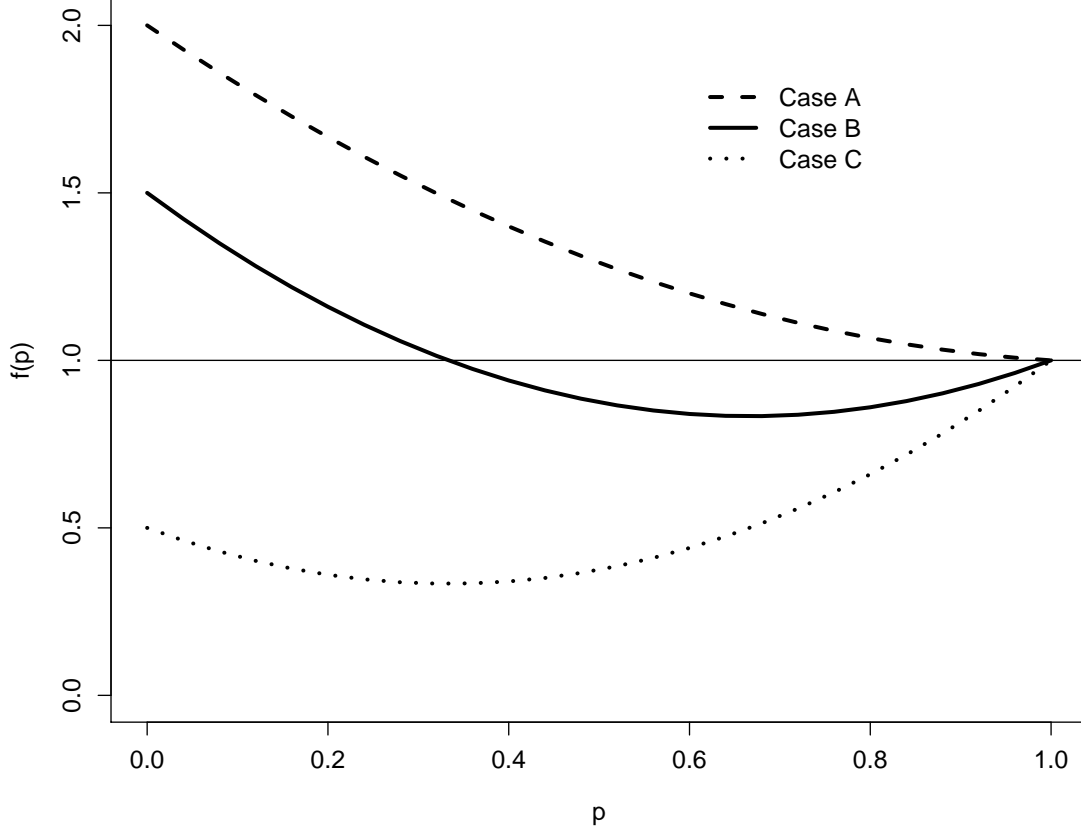
This observation suggests that the information borne by the likelihood ratio is very closely related to the information inherent in the PD curve. More specifically, we obtain the following characterisation of (3.9b) which is basically the likelihood ratio version of (3.5a).

Proposition 3.2 *Let $\pi_x > 0$, $x = 1, \dots, k$ be a probability distribution. Assume that $x \mapsto \lambda(x)$ is a positive function for $x = 1, \dots, k$. Consider the equation*

$$\sum_{x=1}^k \frac{\pi_x}{p + (1 - p) \lambda(x)} = 1. \quad (3.10a)$$

Then with regard to solutions $p \in [0, 1]$ of (3.10a) other than $p = 1$ the following statements hold:

Figure 1: Illustration for the proof of proposition 3.2. The three possibilities for the shape of the graph of the function defined by (3.11a).



- (i) Assume that $x \mapsto \lambda(x)$ is a mapping onto a constant, i.e. $\lambda(x) = \lambda$ for all $x = 1, \dots, k$. Then all $p \in [0, 1]$ are solutions of (3.10a) if $\lambda = 1$ and there is no solution $p \in [0, 1]$ if $\lambda \neq 1$.
- (ii) Assume that $x \mapsto \lambda(x)$ is **not** a mapping onto a constant. Then there exists a solution $p \in [0, 1]$ of (3.10a) if and only if

$$\sum_{x=1}^k \frac{\pi_x}{\lambda(x)} \geq 1 \quad \text{and} \quad \sum_{x=1}^k \pi_x \lambda(x) > 1. \quad (3.10b)$$

If there exists a solution $p \in [0, 1]$ of (3.10a) then this solution is unique. The unique solution is $p = 0$ if and only if

$$\sum_{x=1}^k \frac{\pi_x}{\lambda(x)} = 1. \quad (3.10c)$$

Proof. Statement (i) is obvious. With regard to statement (ii), define the function $f : [0, 1] \rightarrow$

$(0, \infty)$ by

$$f(p) = \sum_{x=1}^k \frac{\pi_x}{p + (1-p)\lambda(x)}. \quad (3.11a)$$

Observe that f is twice continuously differentiable in p with

$$f'(p) = \sum_{x=1}^k \frac{(\lambda(x) - 1)\pi_x}{(p + (1-p)\lambda(x))^2} \quad \text{and} \quad (3.11b)$$

$$f''(p) = 2 \sum_{x=1}^k \frac{(\lambda(x) - 1)^2 \pi_x}{(p + (1-p)\lambda(x))^3}. \quad (3.11c)$$

From (3.11a) and (3.11b) we obtain

$$f(0) = \sum_{x=1}^k \frac{\pi_x}{\lambda(x)}, \quad f(1) = 1, \quad \text{and} \quad f'(1) = \sum_{x=1}^k \pi_x \lambda(x) - 1. \quad (3.11d)$$

(3.11c) implies $f''(p) > 0$ because $\lambda(x)$ is not constant by assumption. Hence f is strictly convex in p . The strict convexity of f implies that the shape of the graph of f is determined by (3.11d) and that only the following three cases can occur:

$$\begin{aligned} \text{case A:} \quad & f(0) < 1 \quad \text{and} \quad f'(1) > 0, \\ \text{case B:} \quad & f(0) \geq 1 \quad \text{and} \quad f'(1) > 0, \quad \text{or} \\ \text{case C:} \quad & f(0) > 1 \quad \text{and} \quad f'(1) \leq 0. \end{aligned}$$

A stylised illustration of the three different possible shapes of the graph of f is shown in figure 1. Only in case B there is a second (and only one) intersection at a $p < 1$ of the horizontal line through 1. By (3.11d), case B is equivalently described by (3.10b). The second intersection of the horizontal line through 1 occurs if and only if $f(0) = 1$ which is equivalent to (3.10c). **q.e.d.**

At first glance, proposition 3.2 might appear as an unnecessarily complicated way to describe the interplay of unconditional rating profile, likelihood ratio, and unconditional PD. However, proposition 3.2 becomes interesting when we try to construct the joint distribution of a borrower's rating X at the beginning of the observation period and the borrower's state S at the end of the period from an unconditional rating profile and a candidate likelihood ratio (which might have been estimated separately). In this context, proposition 3.2 tells us that the construction will work only if condition (3.10b) is satisfied. In contrast, by proposition 3.1 (iii) the construction is always possible if one combines an unconditional rating profile with a candidate PD curve (assuming that all its components take values between 0 and 1).

Actually, from proposition 3.2 it is not yet clear that it gives indeed rise to a fully specified joint distribution of rating X and default or survival state S . This, however, is confirmed by the next proposition.

Proposition 3.3 *Let $\pi_x > 0$, $x = 1, \dots, k$ be a probability distribution. Assume that $x \mapsto \lambda(x)$ is a positive function for $x = 1, \dots, k$ and that equation (3.10a) has a solution $0 < p < 1$. Then there exists a unique joint distribution of X and S such that $x \mapsto \lambda(x)$ is the likelihood ratio associated with the joint distribution in the sense of equation (3.7).*

Table 7: Unconditional default rates and accuracy ratios for the 2009 all corporate and non-financial corporate data from table 2.

Sample	Default rate	Accuracy ratio
All corporates 2009	3.99%	82.7%
Non-financial corporates 2009	5.44%	83.8%

Proof. Define

$$\begin{aligned}
\Pr[D] &= p, \\
\Pr[X = x | D] &= \frac{\pi_x}{p + (1 - p) \lambda(x)}, \quad x = 1, \dots, k, \\
\Pr[X = x | N] &= \frac{\pi_x \lambda(x)}{p + (1 - p) \lambda(x)}, \quad x = 1, \dots, k.
\end{aligned} \tag{3.12}$$

Then both $\Pr[X = x | D]$ and $\Pr[X = x | N]$ are proper probability distributions on $\{1, \dots, k\}$. Hence, by proposition 3.1 there is a unique distribution of X and S such that $\Pr[X = x | D]$ and $\Pr[X = x | N]$ are its conditional rating profiles and p is its unconditional PD. By construction, (3.7) holds. **q.e.d.**

3.3. Estimation period example

In this section, we illustrate the concepts introduced in sections 3.1 and 3.2 by revisiting the S&P data for 2009 presented in section 2. For the sake of a more convenient notation we map the S&P rating symbols CCC-C, B-, B, ..., AA+, AAA onto the numbers $1, \dots, 17$ (hence grade 17 stands for the least risky grade AAA).

Columns 2 and 5 of table 5 show the unconditional rating profiles $x \mapsto \Pr[X = x]$, $\{1, \dots, 17\} \rightarrow [0, 1]$ for the all corporates and non-financial corporates samples respectively.

Table 7 shows the empirical unconditional default rates and accuracy ratios for our estimation data (i.e. the 2009 S&P data). The accuracy ratios were calculated according to (3.6).

Table 8 presents both the empirically observed grade-level default rates and the smoothed PD curves (according to appendix B, with the values from table 7 as targets) for the 2009 S&P data. It is hard to assess directly from the numbers how well or badly the smoothed curves fit the empirical data. Therefore we derive implied default profiles and compare them by means of a χ -squared test to the observed default profiles.

Table 9 shows the empirically observed and implied default profiles for the 2009 S&P data. Equation (3.5b) was applied to calculate the implied profiles. Clearly the fit is worse for the all corporates data. This impression is confirmed by application of the χ -squared test as described in section 2.2. To apply the test, choose the implied profiles as Null hypothesis distributions and test against the default numbers for 2009 tabulated in the upper and lower panels of table 2.

The test results (based on Monte-Carlo approximation) are a p-value of 9% for the fit of the implied all corporates default profile and a p-value of 23% for the fit of the implied non-financial

Table 8: Grade-level default rates and smoothed conditional PDs (PD curves) for the 2009 all corporate and non-financial corporate data from table 2. All numbers in %.

Rating grade	All corporates		Non-financial corporates	
	Default rate	Smoothed PD	Default rate	Smoothed PD
AAA	0.000	0.003	0.000	0.001
AA+	0.000	0.006	0.000	0.002
AA	0.000	0.012	0.000	0.004
AA-	0.000	0.025	0.000	0.009
A+	0.294	0.047	0.000	0.015
A	0.392	0.091	0.000	0.031
A-	0.000	0.173	0.000	0.068
BBB+	0.402	0.299	0.000	0.132
BBB	0.185	0.495	0.000	0.249
BBB-	1.089	0.797	0.000	0.443
BB+	0.000	1.138	0.000	0.673
BB	1.017	1.518	0.833	0.965
BB-	0.907	2.280	1.127	1.583
B+	5.479	3.943	5.141	3.085
B	9.959	7.999	9.862	7.451
B-	17.162	19.557	19.305	21.512
CCC-C	48.421	48.355	51.497	55.396

corporates default profile. Hence the fit for the non-financial corporates is satisfactory while for all corporates the fit could be rejected as too poor at 10% type-I error level. However, given the inversions of default rates in the all corporates data it might be hard to get a much better fit with any other forced monotonic PD curve estimate. We therefore adopt both the all corporate and the non-financial smoothed PD curves from table 8 as starting points for the PD curve calibration examples described in section 4 below.

4. Calibration approaches

The result of the estimation period is a fully specified (and smoothed) model for the joint distribution of a borrower's beginning of the period rating X and end of the period solvency state S . In this section, we discuss how to combine the estimation period model with observations from the beginning of the forecast period in order to predict the grade-level default rates that are observed at the end of the forecast period. This process is often referred to as *calibration of the PD curve*.

Table 9: Empirical and implied default profiles for the 2009 all corporate and non-financial corporate data from table 2. All numbers in %.

	All corporates		Non-financial corporates	
Rating grade	Empirical profile	Implied profile	Empirical profile	Implied profile
AAA	0.0000	0.0010	0.0000	0.0001
AA+	0.0000	0.0009	0.0000	0.0001
AA	0.0000	0.0094	0.0000	0.0012
AA-	0.0000	0.0261	0.0000	0.0027
A+	0.4274	0.0685	0.0000	0.0068
A	0.8547	0.1989	0.0000	0.0330
A-	0.0000	0.4041	0.0000	0.0937
BBB+	0.8547	0.6355	0.0000	0.1942
BBB	0.4274	1.1436	0.0000	0.4495
BBB-	2.1368	1.5642	0.0000	0.6695
BB+	0.0000	1.2934	0.0000	0.6728
BB	1.2821	1.9143	0.9756	1.1292
BB-	1.7094	4.2968	1.9512	2.7405
B+	10.2564	7.3802	9.7561	5.8539
B	20.5128	16.4769	20.9756	15.8468
B-	22.2222	25.3242	24.3902	27.1784
CCC-C	39.3162	39.2628	41.9512	45.1271

Notation. All objects (like probabilities and the likelihood ratio) from the estimation period are labelled with subscript 0. All objects from the forecast period are labelled with subscript 1.

In the following we will make use, in particular, of assumptions on the invariance or specific transformation between estimation and forecast period of

- the conditional rating profiles $\Pr_0[X = x | D]$ and $\Pr_0[X = x | N]$ (for $x = 1, \dots, k$),
- the PD curve $x \mapsto \Pr_0[D | X = x]$, and
- the likelihood ratio $x \mapsto \lambda_0(x)$.

Imagine we are now at the beginning of the forecast period. The borrowers' states of solvency at the end of the period are yet unknown. The objective of the forecast period is to predict the default rates to be observed at the end of the period for the rating grades $1, \dots, k$ by conditional PDs (PD curve) $\Pr_1[D | X = x]$, $x = 1, \dots, k$. There are different forecast techniques for the conditional PDs. The selection of a suitable technique, in particular, depends on what we already know at the beginning of the forecast period about the joint distribution of a borrower's rating X at the beginning of the period and the borrower's solvency state S at the end of the period. We will look in detail at the following two possibilities:

- The unconditional rating profile $\Pr_1[X = x]$, $x = 1, \dots, k$ is known. This is likely to be

the case for a newly developed rating model if all borrowers can be re-rated with the new model in a big-bang effort before the beginning of the forecast period. It will also be the case if an existing rating model is re-calibrated. Even if in the case of a new rating model no timely re-rating of the whole portfolio is feasible, it might still be possible (and should be tried) to re-rate a representative sample of the borrowers in the portfolio such that a reliable estimate of the unconditional rating profile is available. Where this is not possible, the rating model should be used in parallel run with the incumbent rating model until such time as the full rating profile of the portfolio has been determined. Only then a PD curve forecast with some chance of being accurate can be made. This might be one of the reasons for the ‘credible track record’ requirement of the Basel Committee (BCBS, 2006, paragraph 445). However, we will see in section 4.4 that as soon as a forecast of the unconditional PD is given a meaningful if not accurate PD curve forecast can be made without knowledge of the actual unconditional rating profile. This forecast could be used for a preliminary calibration during the Basel II ‘track record’ period.

- *An estimate of the unconditional PD p_1 for the forecast period is available.* This forecast could be a proper best estimate, a pessimistic estimate for stress testing purposes, or a long-run estimate for the purpose of a through-the-cycle (TTC) calibration². How to forecast the unconditional default rate of the portfolio is not a topic of this paper. For to come up with a reliable estimate is actually a major and non-trivial task that might be done by means of a regression of historical portfolio-wide default rates on a suitable set of predictive co-variables. See, for instance, Engelmann and Porath (2012) for such an approach.

These two possibilities are not exclusive nor do they necessarily occur together. That is why, in the following, we discuss four cases:

- **Case 1.** Neither the unconditional rating profile nor the unconditional PD for the forecast period are known.
- **Case 2.** The unconditional rating profile for the forecast period is known but no independent estimate of the unconditional PD is available.
- **Case 3.** The unconditional rating profile for the forecast period is not known but an independent estimate of the unconditional PD is available.
- **Case 4.** The unconditional rating profile for the forecast period is known and an independent estimate of the unconditional PD is available.

For each of the four cases we will present one or more techniques to forecast a set of model components needed to specify a full model. See proposition 3.1 for the main possibilities to specify a full proper one-period model of borrower’s beginning of the period rating and end of the period solvency state. We will illustrate the forecast techniques presented with numerical examples based on the S&P data from table 2. In none of the four cases there is sufficient information from the forecast period available to completely specify a model. That is why assumptions about inter-period invariance of model components play an important role in the forecast process.

²See Heitfield (2005) for a detailed discussion of point-in-time (PIT) and TTC rating and PD estimation approaches. See Löffler (2012) for the question of how much TTC agency ratings are.

4.1. Invariance assumptions

Forecasting without assuming that some of the features observed in the estimation period are invariant (i.e. unchanged) between the estimation and forecast periods is impossible. Ideally, any assumption of invariance should be theoretically sound, and it should be possible to verify it by back-testing. In this section, we briefly discuss which invariance assumptions for the one-period model from section 3 we should closer look at in this section.

- It is obvious that no invariance assumptions must be made on objects that can be observed or reliably estimated in a separate forecast exercise at the beginning of the forecast period:
 - As explained above, in particular, the actual unconditional rating profile of the portfolio should be known at the beginning of the observation period.
 - Although we will see in section 4.3 that under strong enough invariance assumptions a meaningful forecast of the unconditional default rate can be extracted from the actual unconditional rating profile it is strongly recommended not to rely on such a forecast but to try and independently forecast the unconditional default rate.
- As the future solvency states of the borrowers in the portfolio are not yet known at the beginning of the forecast period, assuming that both conditional rating profiles are invariant could make sense. However, this assumption is likely to make for inconsistency issues due to equation (3.4a) in case of a known actual unconditional rating profile.
- Assuming that the likelihood ratio is invariant is less restrictive than the assumption of invariant conditional rating profiles. It turns out, however, that if the actual unconditional rating profile is known the likelihood ratio invariance assumption implies a forecast of the unconditional default rate that might be in conflict with other forecasts of the unconditional default rate.
- Instead of assuming that both conditional rating profiles are invariant, one could also assume that only one of the two is invariant. In case of the assumption of an invariant survival rating profile this turns out to be impracticable because too often no proper model will be the result. That is why we do not discuss the details of this invariance assumption in this paper.
- Assuming the default rating profile as invariant is a much more promising approach. As a matter of fact we will see that for new models it might even be the only logically consistent approach.
- Invariance assumptions might be weakened by restating them as shape invariance assumptions.
 - For instance, a common approach is to assume that the shape of the PD curve is preserved between the estimation and the forecast periods. This can be accomplished by scaling the PD curve with a constant multiplier that is determined at the beginning of the forecast period (see, e.g., [Falkenstein et al., 2000](#), page 67). (3.4b) shows that the scaled PD curve strongly depends on the estimation period unconditional PD. Hence making use of the scaled PD curve for forecasts in the forecast period might ‘contaminate’ the forecast with the estimation period unconditional PD which might

be quite different from the forecast period unconditional PD. We include the scaled PD curve approach nonetheless in the subsequent more detailed discussion because of its simplicity and popularity.

- Scaling the likelihood ratio instead of the PD curve avoids the contamination issue we have observed for the scaled PD curve. The price to pay for this is that this approach is less straightforward than the scaled PD curve approach.

4.2. Case 1: No unconditional rating profile and no unconditional PD given

From a risk management point of view it is undesirable to have no current data at all. In a stable economic environment, this approach might be justifiable nonetheless. One could assume that the model from the estimation period works without any adaptations also for the forecast period. Of course, at the end of the forecast period, we can then back-test the default profile from the observation period against the grade-level default frequencies observed. Formally, the assumption made in case 1 may be described as

$$\Pr_1[X = x] = \Pr_0[X = x], \quad x = 1, \dots, k, \quad (4.1)$$

$$\Pr_1[D | X = x] = \Pr_0[D | X = x], \quad x = 1, \dots, k. \quad (4.2)$$

Table 3, combined with table 8, indicates that it would not have been a good idea to try and predict the grade-level S&P default rates of 2010 and 2011 with the PD curve from 2009. Alternatively, one might try and come up with plausible assumptions on the forecast period unconditional rating profile or unconditional PD – which would bring us into case 2, case 3, or case 4.

4.3. Case 2: Unconditional rating profile but no unconditional PD given

In this case, the unconditional rating profile $\Pr_1[X = x]$, $x = 1, \dots, k$ can directly be observed at the beginning of the forecast period. We consider three approaches to prediction in the forecast period that may lead to proper models for the forecast period:

- Assume that the PD curve is invariant, i.e. (4.2) holds.
- Assume that both conditional rating profiles are invariant, i.e.

$$\begin{aligned} \Pr_1[X = x | D] &= \Pr_0[X = x | D], \quad x = 1, \dots, k, \quad \text{and} \\ \Pr_1[X = x | N] &= \Pr_0[X = x | N], \quad x = 1, \dots, k. \end{aligned} \quad (4.3)$$

- Assume that the likelihood ratio is invariant, i.e.

$$\lambda_1(x) = \lambda_0(x), \quad x = 1, \dots, k. \quad (4.4)$$

Note that (4.4) is implied by (4.3) and, hence, is a weaker assumption.

Assumption (4.2). As mentioned in section 4.1, it might not be the best idea to work under this assumption because there is a risk to ‘contaminate’ the forecast with the estimation period unconditional default rate p_0 . However, by proposition 3.1 the combination of unconditional rating profile with any PD curve creates a unique proper model of a borrower’s beginning of the period rating and end of the period state of solvency. In particular, by (3.5a) this approach implies a forecast of the unconditional default rate in the forecast period:

$$p_1 = \sum_{x=1}^k \Pr_0[D | X = x] \Pr_1[X = x]. \quad (4.5)$$

Assumption (4.3). Equation (3.4a) functions here as a constraint. The unknown unconditional PD p_1 and the two conditional profiles therefore must satisfy

$$\Pr_1[X = x] = p_1 \Pr_0[X = x | D] + (1 - p_1) \Pr_0[X = x | N], \quad x = 1, \dots, k. \quad (4.6a)$$

Hence, as all three profiles $\Pr_1[X = x]$, $\Pr_0[X = x | D]$, and $\Pr_0[X = x | N]$ are known, we have k equations for the one unknown p_1 . In general, it seems unlikely that all the k equations can be simultaneously satisfied if only one variable can be freely chosen. However, we can try and compute a best fit by solving the following least squares optimisation problem:

$$\begin{aligned} p_1^* &= \arg \min_{p_1 \in [0,1]} \sum_{x=1}^k (\Pr_1[X = x] - p_1 \Pr_0[X = x | D] - (1 - p_1) \Pr_0[X = x | N])^2 \\ \Rightarrow p_1^* &= \sum_{x=1}^k \frac{(\Pr_1[X = x] - \Pr_0[X = x | N]) (\Pr_0[X = x | D] - \Pr_0[X = x | N])}{(\Pr_0[X = x | D] - \Pr_0[X = x | N])^2}. \end{aligned} \quad (4.6b)$$

Observation (4.6b) is interesting because it indicates a technique to extract a forecast of the unconditional PD from the unconditional rating profile at the beginning of the forecast period that also avoids the contamination issue observed for the invariant PD curve. It should be checked whether the forecast PD p_1^* is indeed in line with the profile $x \mapsto \Pr_1[X = x]$. This can readily be done because with p_1^* from (4.6b) we obtain an implied unconditional rating profile

$$\Pr_1^*[X = x] = p_1^* \Pr_0[X = x | D] + (1 - p_1^*) \Pr_0[X = x | N], \quad x = 1, \dots, k. \quad (4.7)$$

This can be χ^2 -tested against the grade-level frequencies of borrowers at the beginning of the forecast period. If the hypothesis that $x \mapsto \Pr_1[X = x]$ is just a random realisation of $x \mapsto \Pr_1^*[X = x]$ cannot be rejected we can proceed to predict the PD curve on the basis of $x \mapsto \Pr_1^*[X = x]$ by using (3.5b):

$$\Pr_1[D | X = x] = \frac{p_1^* \Pr_0[X = x | D]}{\Pr_1^*[X = x]}, \quad x = 1, \dots, k. \quad (4.8)$$

The optimisation problem (4.6b) is convenient for deriving a forecast of p_1 from the unconditional rating profile because it yields a closed-form solution. In principle, there is no reason why the least squares should not be replaced with a – say – least absolute value optimisation. This would result in a slightly different forecast of p_1 . However, as we will check the appropriateness of the p_1 forecast by applying a χ -squared test as described in section 2.2, it seems natural to also look at the variant of (4.6b) where the χ -squared statistic is directly minimised. Proposition C.1 in appendix C shows that this minimisation problem is well-posed and has a unique solution.

Table 10: Forecasts of 2010 and 2011 S&P unconditional default rates and p-values for goodness of fit tests of 2010 and 2011 unconditional rating profiles. The forecast approaches are described in section 4.3.

Forecast for	2010			
Sample	All corporates		Non-financial corporates	
Observed default rate	1.14%		1.43%	
	Forecast DR	p-value	Forecast DR	p-value
Invariant PDs (4.2)	4.32%	Exact	5.96%	Exact
Least squares (4.6b)	4.80%	0.0037	6.83%	0.3338
Least χ^2 (C.3)	5.35%	0.0051	7.17%	0.3467
Invariant LR (4.4)	5.38%	Exact	7.19%	Exact
Forecast for	2011			
Sample	All corporates		Non-financial corporates	
Observed default rate	0.75%		0.95%	
	Forecast DR	p-value	Forecast	p-value
Invariant PDs (4.2)	3.75%	Exact	4.87%	Exact
Least squares (4.6b)	3.50%	$< 10^{-10}$	3.60%	1.5×10^{-5}
Least χ^2 (C.3)	2.84%	$< 10^{-10}$	3.26%	1.6×10^{-5}
Invariant LR (4.4)	2.79%	Exact	3.27%	Exact

Assumption (4.4). Like for assumption (4.3), it is not a priori clear that a proper model of a borrower's rating profile and solvency state can be based on the unconditional profile $x \mapsto \Pr_1[X = x]$ and the likelihood ratio $\lambda_0(x)$. The necessary and sufficient condition for the likelihood ratio to match the rating profile is provided in equation (3.10b) of proposition 3.2, with $\pi_x = \Pr_1[X = x]$ and $\lambda(x) = \lambda_0(x)$.

If condition (3.10b) is satisfied then proposition 3.3 implies that there is a unique model of a borrower's rating and solvency state with characteristics $\Pr_1[X = x]$ and $\lambda_0(x)$. The unconditional PD in this model is determined as the unique solution p_1 of equation (3.10a), and we can calculate the PD curve by (3.8a):

$$\Pr_1[D | X = x] = \frac{p_1}{p_1 + (1 - p_1) \lambda_0(x)}, \quad x = 1, \dots, k. \quad (4.9)$$

Summary. Table 10 displays some forecast results that were calculated with the approaches described in this section. Forecast values for the 2010 and 2011 S&P unconditional default rates are presented together with assessments of the goodness of fit of the actual unconditional rating profiles by the implied or assumed unconditional rating profiles. It is immediately clear from table 10 that the forecasts of the unconditional default rates are much too high in all cases. That is why we did not bother to show the grade-level forecast default rates or any other model

Table 11: S&P grade-level smoothed likelihood ratios (defined by (3.7)) for corporates in 2009, 2010 and 2011. Source: Own calculations.

Rating grade	All corporates			Non-financial corporates		
	2009	2010	2011	2009	2010	2011
AAA	1,501.55	53,720.74	20,221.19	6,813.88	20,028,740.00	131,709.90
AA+	715.28	20,477.36	7,734.58	3,017.50	5,562,260.00	56,168.55
AA	353.69	8,583.13	3,410.45	1,304.81	1,253,332.00	21,913.78
AA-	167.12	3,099.52	1,365.00	646.25	367,937.00	9,281.64
A+	88.18	1,161.49	547.00	372.94	138,158.70	4,404.54
A	45.53	436.37	227.20	182.81	38,238.02	1,645.46
A-	23.98	177.13	100.78	84.70	9,612.43	601.66
BBB+	13.89	83.35	51.03	43.44	2,988.50	255.66
BBB	8.37	40.00	27.65	23.04	945.70	114.38
BBB-	5.17	19.85	15.00	12.94	335.61	53.48
BB+	3.61	12.23	9.58	8.49	168.65	31.20
BB	2.70	8.28	6.74	5.91	93.27	20.33
BB-	1.78	4.93	4.24	3.58	42.88	11.52
B+	1.01	2.46	2.19	1.81	15.21	5.07
B	0.48	0.96	0.79	0.71	3.59	1.44
B-	0.17	0.27	0.20	0.21	0.57	0.28
CCC-C	0.04	0.05	0.04	0.05	0.05	0.04

characteristics as the fit would have been equally poor.

We have argued above that assumption (4.2) is suboptimal for risking undesirable impact on the forecast of the estimation period unconditional default rate. Assumption (4.4) is the most promising of the three assumptions we have explored because it guarantees exact fit of the unconditional rating profile and avoids contamination of the forecast. Assumption (4.3) is stronger than (4.4) because it implies (4.4). In principle, assumption (4.3) will hardly ever provide a proper model because it is rather unlikely that that the overdetermined equation (4.6a) has an exact solution. By (4.6b) or proposition C.1, however, we can determine an approximate fit that could turn out to be statistically indistinguishable from the rating profile at the beginning of the forecast period – which would make assumption (4.3) a viable approach, too. Indeed, according to table 10 this is the case for the 2010 forecast for the non-financial corporates.

As ‘contamination’ by the 2009 unconditional default rate is prevented under assumptions (4.3) and (4.4), it is interesting to speculate why the implied default rate forecasts are so poor nonetheless. The natural conclusion is that the assumptions are simply wrong for the S&P data. Indeed, as table 11 demonstrates for the likelihood ratio, with hindsight it is clear that the invariance assumptions made in this sections do not hold. An alternative and complementary explanation could however be that the S&P ratings made in 2009 and 2010 are over-pessimistic and for this reason generate too high default rate forecasts. This explanation is supported by the observation

that in 2009 the downgrade-to-upgrade ratio for the S&P corporate ratings was 3.99 (S&P, 2012, Table 6) – which could presumably not even be compensated by the 2010 downgrade-to-upgrade ratio of 0.74. Possibly, the truth is a mixture of these two explanations.

4.4. Case 3: No unconditional rating profile but unconditional PD given

In this case, we assume that a forecast unconditional PD $0 < p_1 < 1$ is given that is likely to differ from the estimation period unconditional PD. But the unconditional current rating profile is assumed not to be known. This would typically be the case if a rating model was newly developed and it was not possible to rate all the borrowers in the portfolio in one big-bang effort. The new ratings would then only become available in the course of the regular annual rating process. This is clearly suboptimal, in particular with a view on the validation of the new rating model, but sometimes unavoidable due to limitation of resources.

In this situation, proposition 3.1 suggests assumption (4.3) as the only possibility to infer a full model of a borrower's beginning of the period rating and end of the period state of solvency. If, however, it is sufficient to obtain an estimate of the forecast period PD curve then it is solely the estimation period likelihood ratio $x \mapsto \lambda_0(x)$ that one needs to know in addition to the unconditional PD p_1 . This follows from equation (4.9). Elkan (2001, theorem 2) stated this observation as ‘change in base rate’ theorem.

4.5. Case 4: Unconditional rating profile and unconditional PD given

In this case, it is assumed that the unconditional rating profile $\Pr_1[X = x]$, $x = 1, \dots, k$ can directly be observed at the beginning of the forecast period, and it is also assumed that a forecast unconditional PD $0 < p_1 < 1$ is given that is likely to differ from the estimation period unconditional PD. Like in case 2, there are several approaches to prediction in the forecast period that may lead to proper models for the forecast period:

- Assume that the default rating profile is invariant, i.e.

$$\Pr_1[X = x | D] = \Pr_0[X = x | D], \quad x = 1, \dots, k. \quad (4.10)$$

- Assume that the discriminatory power of the model as measured by the accuracy ratio (see (3.6)) is invariant, i.e.

$$\text{AR}_1 = \text{AR}_0. \quad (4.11)$$

Note that (4.11) is a weaker assumption than (4.3) because (4.11) is implied by (4.3).

- Assume that the estimation period PD curve can be linearly scaled to become the forecast period PD curve, i.e. there is a constant $c_{\text{PD}} > 0$ such that

$$\Pr_1[D | X = x] = c_{\text{PD}} \Pr_0[D | X = x], \quad x = 1, \dots, k. \quad (4.12)$$

- Assume that the estimation period likelihood ratio can be linearly scaled to become the forecast period likelihood ratio, i.e. there is a constant $c_{\text{LR}} > 0$ such that

$$\lambda_1(x) = c_{\text{LR}} \lambda_0(x), \quad x = 1, \dots, k. \quad (4.13)$$

In principle, a fifth approach is cogitable, namely to assume that the survivor rating profile does not change from the estimation period to the forecast period. However, it is very unlikely that this approach results in a proper forecast period model with a proper default rating profile. That is why we do not discuss this approach.

Assumption (4.10). This assumption is not necessarily viable as again (3.4a) must be satisfied. It turns out that assumption (4.10) makes for a proper model of a borrower's rating and state of solvency if and only if we have for all $x = 1, \dots, k$

$$p_1 \Pr_0[X = x | D] \leq \Pr_1[X = x] \quad \text{and} \quad p_1 (1 - \Pr_0[X = x | D]) \leq (1 - \Pr_1[X = x]). \quad (4.14a)$$

If (4.14a) holds then by (3.5b) we obtain the following equation for the PD curve:

$$\Pr_1[D | X = x] = \frac{p_1 \Pr_0[X = x | D]}{\Pr_1[X = x]}. \quad (4.14b)$$

Actually, there are two slightly different approaches to implement assumption (4.10):

- (i) Use a smoothed version of the estimation period default profile that could be derived via equation (3.5b) from a smoothed PD curve – which in turn might have been determined by QMM as described in section 3.1.
- (ii) Use the observed estimation period default profile and the given forecast period unconditional profile to determine by means of (3.5c) an implied raw survivor profile. Based on this survivor profile and the observed estimation period default profile deploy equation (3.6) to compute a forecast accuracy ratio. Apply then QMM as described in section 3.1 to determine a smoothed PD curve for the forecast period.

Compared to approach (i), approach (ii) has the advantage of always delivering a monotonic PD curve. That is why for the purpose of this paper we implement assumption (4.10) in the shape of (ii) although anecdotal evidence shows that the performance of (ii) is not necessarily better than the performance of (i).

Assumption (4.11). Actually, even with unconditional rating profile, unconditional PD, and accuracy ratio given the joint distribution of a borrower's beginning of the period rating and end of the period state is not uniquely determined. We suggest applying QMM as in the estimation period (see section 3.3) and described in appendix B to compute a PD curve as a forecast of the grade-level default rates. There is, however, the problem that QMM requires the rating profile conditional on survival as an input – which cannot be observed or implied at this stage. But QMM is fairly robust with regard to the coding of the rating grades used as input to the algorithm. That is why approximating the rating profile conditional on survival with the unconditional rating profile (known by assumption) seems to work well.

Assumption (4.12). The constant c_{PD} is determined by equation (3.5a):

$$c_{PD} = \frac{p_1}{\sum_{x=1}^k \Pr_0[D | X = x] \Pr_1[X = x]} \quad (4.15a)$$

Table 12: 2010 and 2011 grade-level forecast default rates for S&P all corporates ratings. P-values are for the χ^2 -tests of the implied default profiles. All values in %.

	Default rate	Invariant default profile (4.10)	Invariant AR (4.11)	Scaled PDs (4.12)	Scaled likelihood ratio (4.13)
	2010: Unconditional default rate 1.141				
AAA	0	0.0013	0.0004	0.0007	0.0005
AA+	0	0.0025	0.0009	0.0015	0.0011
AA	0	0.0048	0.0018	0.0031	0.0023
AA-	0	0.0097	0.0040	0.0066	0.0049
A+	0	0.0177	0.0086	0.0125	0.0093
A	0	0.0327	0.0183	0.0241	0.018
A-	0	0.0591	0.0366	0.0458	0.0342
BBB+	0	0.0977	0.0652	0.0789	0.059
BBB	0	0.1555	0.1145	0.1307	0.0979
BBB-	0	0.2409	0.1955	0.2107	0.1581
BB+	0.7874	0.3332	0.2827	0.3006	0.2263
BB	0.3623	0.4328	0.3806	0.4012	0.3029
BB-	0.5277	0.6243	0.5631	0.6024	0.4576
B+	0	1.0252	0.9460	1.0417	0.8023
B	0.6881	1.9704	1.8843	2.1134	1.6844
B-	2.069	4.605	4.5164	5.1671	4.5716
CCC-C	22.2727	13.4867	14.5179	12.7755	15.576
P-value	Exact	5.34	7.9	4.1	10.5
	2011: Unconditional default rate 0.752				
AAA	0	0.0004	0.0003	0.0006	0.0004
AA+	0	0.0009	0.0006	0.0012	0.0009
AA	0	0.0019	0.0013	0.0024	0.0018
AA-	0	0.0041	0.0027	0.005	0.0039
A+	0	0.0079	0.0058	0.0095	0.0074
A	0	0.0155	0.0120	0.0183	0.0143
A-	0	0.0298	0.0236	0.0347	0.0271
BBB+	0	0.0519	0.0416	0.0599	0.0468
BBB	0	0.0867	0.0691	0.0992	0.0777
BBB-	0.1969	0.1404	0.1147	0.16	0.1256
BB+	0	0.2007	0.1662	0.2282	0.1797
BB	0	0.2677	0.2219	0.3046	0.2405
BB-	0	0.4006	0.3248	0.4573	0.3635
B+	0.3929	0.6913	0.5567	0.7909	0.6378
B	1.1945	1.4207	1.2710	1.6046	1.3415
B-	3.9867	3.6221	3.7941	3.9231	3.6627
CCC-C	15.942	11.9924	13.3871	9.6998	12.7721
P-value	Exact	71.9	89.68	37.65	82.38

However, if $c_{PD} > 1$ the resulting model could be improper because by (4.12) it could turn out that $\Pr_1[D | X = x] > 1$ for some x . If the resulting model under assumption (4.12) is proper the implied default profile is as follows:

$$\Pr_1[X = x | D] = c_{PD} \Pr_0[D | X = x] \Pr_1[X = x] / p_1, \quad x = 1, \dots, k. \quad (4.15b)$$

Assumption (4.13). By (3.9b) we obtain an equation that determines the constant c_{LR} :

$$1 = \sum_{x=1}^k \frac{\Pr_1[X = x]}{p_1 + (1 - p_1) c_{LR} \lambda_0(x)}. \quad (4.16a)$$

Note that

$$\lim_{c \rightarrow \infty} \sum_{x=1}^k \frac{\Pr_1[X = x]}{p_1 + (1 - p_1) c \lambda_0(x)} = 0 \quad \text{and} \quad \lim_{c \rightarrow 0} \sum_{x=1}^k \frac{\Pr_1[X = x]}{p_1 + (1 - p_1) c \lambda_0(x)} = 1/p_1 > 1.$$

Hence equation (4.16a) has always a unique solution $c_{LR} > 0$. By proposition 3.3 then we know that under assumption (4.13) we have a proper model of a borrower's rating and default state. In addition, by proposition 3.2 the resulting forecast likelihood ratio $\lambda_1(x) = c_{LR} \lambda_0(x)$ satisfies the inequalities

$$\sum_{x=1}^k \frac{\Pr_1[X = x]}{\lambda_1(x)} > 1 \quad \text{and} \quad \sum_{x=1}^k \Pr_1[X = x] \lambda_1(x) > 1.$$

This implies the following inequality for c_{LR} :

$$\frac{1}{\sum_{x=1}^k \Pr_1[X = x] \lambda_0(x)} < c_{LR} < \sum_{x=1}^k \frac{\Pr_1[X = x]}{\lambda_0(x)}. \quad (4.16b)$$

(4.16b) is useful because it provides initial values for the numerical solution of (4.16a) for c_{LR} . Once c_{LR} has been determined (3.9a) and (3.8a) imply the following equations for the default profile and the PD curve under assumption (4.13):

$$\Pr_1[X = x | D] = \frac{\Pr_1[X = x]}{p_1 + (1 - p_1) c_{LR} \lambda_0(x)}, \quad x = 1, \dots, k, \quad (4.16c)$$

$$\Pr_1[D | X = x] = \frac{p_1}{p_1 + (1 - p_1) c_{LR} \lambda_0(x)}, \quad x = 1, \dots, k. \quad (4.16d)$$

Summary. Tables 12 and 1 show the results of an application of the approaches presented above to forecasting the 2010 and 2011 grade-level default rates of the S&P all corporates and non-financial corporates portfolios, based on estimates made with data from 2009. To allow for a fair performance comparison, we have made use of prophetic estimates of the 2010 and 2011 unconditional default rates, by setting the value of p_1 to the observed unconditional default rate of the respective year and sample.

In order to express the performance of the different approaches in one number for each approach, we have used the forecast PD curves to derive forecast default profiles by means of (3.5b). The

forecast default profiles can be χ -squared tested against the observed grade-level default numbers from table 2. The p-values of these tests are shown in the last rows of the panels of table 12 and 1 respectively. Recall that higher p-values mean better goodness of fit.

Tables 12 and 1 hence indicate that under the constraints of this section (unconditional rating profile and default rate are given) the scaled likelihood ratio approach (4.13) and the invariant accuracy ratio approach (4.11) work best, followed by the invariant default profile approach (4.10). This observation, however, does not allow an unconditional conclusion that ‘scaled likelihood ratio’ or ‘invariant accuracy ratio’ are the best approaches to PD curve calibration.

In particular, when a new rating model is developed one has often to combine data from several observation periods in order to create a sufficiently large training sample. Estimating the likelihood ratio from such a combined sample would implicitly be based on the assumption of an invariant likelihood ratio. Hence it would be strange to modify the likelihood ratio via scaling in the forecast period. This consistency issue is obviously avoided with the ‘invariant default profile’ and the ‘invariant accuracy ratio’ approaches. As we have seen, to implement the ‘invariant accuracy ratio’ approach we need to approximate the forecast period survivor profile by the forecast period unconditional rating profile. This approximation could be poor if the forecast period unconditional default rate is high. Hence, depending on what approach had been followed in the estimation period and how big the forecast period unconditional default rate is, the ‘invariant default profile’ approach (4.10) could be preferable for the forecast period despite its only moderate performance in our numerical examples.

5. Conclusions

Accurate (re-)calibration of a rating model requires careful consideration of a number of questions that include, in particular, the question of which model components can be assumed to be invariant between the estimation period of the model and the forecast period. Looking at PD curve calibration as a problem of forecasting rating-grade level default rates, we have discussed a one-period model framework that is suitable for the description of a variety of different forecasting techniques.

We have then proceeded to present a number of PD curve calibration techniques and explore the conditions under which the techniques are fit for purpose. We have tested the techniques introduced by applying them to a publicly available dataset of S&P rating and default statistics that can be considered typical for the scope of application of the techniques.

One negative and one positive finding are the main results of our considerations:

- The popular technique of ‘scaling the PD curve’ in order to (re-)calibrate a rating model to a different target unconditional PD is likely to deliver suboptimal results because it implicitly mixes up the unconditional PD of the estimation period and the target PD.
- As shown by example, two techniques that avoid mixing up the unconditional PDs from the estimation and the forecast periods are promising alternatives to ‘scaling the PD curve’. These techniques are ‘scaling the likelihood ratio’ and quasi moment matching of target PD and estimation period accuracy ratio.

A. Appendix: The binormal model with equal variances

In this paper, we apply quasi moment matching (QMM) as suggested by Tasche (2009) for the smoothing of PD curves. QMM requires the numerical solution of a two-dimensional system of non-linear equations. The solution of such an equation system in general is much facilitated if a meaningful initial guess of the solution can be provided. The binormal model we discuss in the following delivers such a guess. In addition, the binormal model provides the main motivation of the QMM approach.

Formally, the binormal model with equal variances is based on the following assumption.

Assumption A.1 *X denotes the continuous score of a borrower at the beginning of the observation period.*

- *The distribution of X conditional on the event D (the borrower defaults during the observation period) is normal with mean μ_D and variance $\sigma^2 > 0$.*
- *The distribution of X conditional on the event N (the borrower remains solvent during the whole observation period) is normal with mean $\mu_N > \mu_D$ and variance $\sigma^2 > 0$.*
- *$p \in (0, 1)$ is the borrower's unconditional PD (i.e. the unconditional probability that the borrower defaults during the observation period).*

Denote by f_D and f_N respectively the conditional densities of the binormal score X . Hence by assumption A.1 we have

$$\begin{aligned} f_D(x) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x - \mu_D)^2}{2\sigma^2}\right), \\ f_N(x) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x - \mu_N)^2}{2\sigma^2}\right). \end{aligned} \tag{A.1}$$

In the continuous case specified by assumption A.1, Bayes' formula implies a PD curve $x \mapsto \Pr[D | X = x]$ similar to the discrete formula (3.4b):

$$\Pr[D | X = x] = \frac{p f_D(x)}{p f_D(x) + (1 - p) f_N(x)} \tag{A.2a}$$

$$= \frac{1}{1 + \exp(\alpha + \beta x)}, \tag{A.2b}$$

$$\alpha = \frac{\mu_D^2 - \mu_N^2}{2\sigma^2} + \log\left(\frac{1 - p}{p}\right), \tag{A.2c}$$

$$\beta = \frac{\mu_N - \mu_D}{\sigma^2}. \tag{A.2d}$$

Note that from (A.2b) it follows that

$$\frac{d \Pr[D | X = x]}{dx} = -\beta \Pr[D | X = x] (1 - \Pr[D | X = x]). \tag{A.3}$$

Hence the absolute value of the slope of the PD curve (A.2b) attains its maximum if and only if $\Pr[D | X = x] = 1/2$ and then the maximum absolute slope is $\beta/4$.

Denote by X_D and X_N independent random variables with $X_D \sim \mathcal{N}(\mu_D, \sigma)$ and $X_N \sim \mathcal{N}(\mu_N, \sigma)$. Then, under assumption A.1, we also obtain a simple formula³ for the discriminatory power of the score X if it is measured as *accuracy ratio* (see, for instance, Tasche, 2009, section 3.1.1):

$$\text{AR} = \Pr[X_D < X_N] - \Pr[X_D > X_N] \quad (\text{A.4a})$$

$$= 2 \Phi \left(\frac{\mu_N - \mu_D}{\sqrt{2} \sigma} \right) - 1. \quad (\text{A.4b})$$

In addition, it is easy to show how the unconditional mean μ and variance τ^2 of the score X can be described in terms of the means and variances of X conditional on default and survival respectively:

$$\mu = p \mu_D + (1 - p) \mu_N, \quad (\text{A.5a})$$

$$\tau^2 = \sigma^2 + p(1 - p)(\mu_D - \mu_N)^2. \quad (\text{A.5b})$$

A close inspection of equations (A.4b), (A.5a) and (A.5b) shows that the conditional variance σ^2 and the conditional means μ_D and μ_N can be written as functions of the unconditional mean, the unconditional variance and the accuracy ratio:

$$\begin{aligned} c &= \sqrt{2} \Phi^{-1} \left(\frac{\text{AR} + 1}{2} \right), \\ \sigma^2 &= \frac{\tau^2}{1 + p(1 - p)c^2}, \\ \mu_N &= \mu + p \sigma c, \\ \mu_D &= \mu - (1 - p) \sigma c. \end{aligned} \quad (\text{A.6})$$

From this, it follows by (A.2c) and (A.2d) that also the coefficients α and β in (A.2b) can be represented in terms of the unconditional mean μ of X , the unconditional variance τ^2 of X , and the discriminatory power AR of X . In particular, we have the following representation of β in terms of the accuracy ratio and the dispersion of the conditional score distributions:

$$\beta = \frac{\sqrt{2} \Phi^{-1} \left(\frac{\text{AR} + 1}{2} \right)}{\sigma}. \quad (\text{A.7})$$

These observations suggest the following three steps approach to identifying initial values for the QMM approach to PD curve smoothing:

- 1) Calculate the mean μ and the standard deviation τ of the unconditional rating profile.
- 2) Use μ and τ together with the unconditional PD p and the accuracy ratio AR implied by the rating profile and the observed grade-level default rates to calculate the conditional standard deviation σ and the conditional means μ_D and μ_N according to (A.6).
- 3) Use equations (A.2c) and (A.2d) to determine initial values for α and β .

The initial values found by this approach will be the closer to the true values, the closer the conditional rating profiles are to normal distributions.

³ Φ denotes the standard normal distribution function $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-1/2 y^2} dy$.

B. Appendix: Quasi moment matching

Equation (A.2a) shows that the unconditional PD has a direct primary impact on the level of the PD curve. Equation (A.7) suggests that the AR of a rating model has a similar impact on the maximum slope of the PD curve. The two observations together suggest that in general a two-parameter PD curve can be fitted to match given unconditional PD and AR.

It may be argued that for a suitably developed rating model based on carefully selected risk factors, the associated PD curve must be monotonic for economic reasons. Under the assumption that the PD curve is monotonic, Tasche (2009, section 5.2) suggested the following robust version of the logistic curve (A.2b) for fitting the PD curve:

$$\begin{aligned} \mathbb{P}[D | X = x] &\approx \frac{1}{1 + \exp(\alpha + \beta \Phi^{-1}(F_N(x)))}, \\ F_N(x) &= \mathbb{P}[X \leq x | N]. \end{aligned} \quad (\text{B.1})$$

The term $\Phi^{-1}(F_N(x))$ in (B.1) transforms the in general non-normal distribution of the ratings conditional on survival into another distribution that is approximately normal even if the rating distribution is not continuous. However, in the discontinuous case $F_N(x) = 1$ may occur which would entail $\mathbb{P}[D | X = x] = 0$. A suitable work-around to avoid this is to replace the distribution function F_N by the average \tilde{F}_N of F_N and its left-continuous version:

$$\tilde{F}_N(x) = \frac{\mathbb{P}[X < x | N] + \mathbb{P}[X \leq x | N]}{2}. \quad (\text{B.2})$$

Define, in addition to F_N , the distribution function F_D of the rating variable X conditional on default by

$$F_D(x) = \mathbb{P}[X \leq x | D]. \quad (\text{B.3})$$

and denote by X_D and X_N independent random variables that are distributed according to F_D and F_N respectively.

For quantifying discriminatory power, we apply again the notion of *accuracy ratio* (AR) as specified in Tasche (2009, eq. (3.28b)):

$$\begin{aligned} \text{AR} &= 2 \mathbb{P}[X_D < X_N] + \mathbb{P}[X_D = X_N] - 1 \\ &= \mathbb{P}[X_D < X_N] - \mathbb{P}[X_D > X_N] \\ &= \int \mathbb{P}[X < x | D] dF_N(x) - \int \mathbb{P}[X < x | N] dF_D(x). \end{aligned} \quad (\text{B.4a})$$

See Hand and Till (2001, section 2) for a discussion of why this definition of accuracy ratio (or the related definition of the area under the ROC curve) is more expedient than the also common definition in geometric terms. Definition (B.4a) of AR takes an ‘ex post’ perspective by assuming the obligors’ states D or N at the end of the observation period are known and hence can be used for estimating the conditional (on default and survival respectively) rating distributions F_D and F_N .

In the case where X is realised as one of a finite number of rating grades $x = 1, \dots, k$, the

accuracy ratio can be calculated from the PD curve as follows:

$$\begin{aligned} \text{AR} = & \frac{1}{p(1-p)} \left(2 \sum_{x=1}^k (1 - P[D | X = x]) P[X = x] \sum_{t=1}^{x-1} P[D | X = t] P[X = t] \right. \\ & \left. + \sum_{x=1}^k P[D | X = x] (1 - P[D | X = x]) P[X = x]^2 \right) - 1. \end{aligned} \quad (\text{B.4b})$$

Where p stands for the unconditional PD as described in (3.2).

Then **quasi moment matching**, for the purpose of this paper means the following procedure:

- 1) Fix target values p_{target} and $\text{AR}_{\text{target}}$ for the unconditional portfolio PD and the accuracy ratio of the rating model.
- 2) Substitute p_{target} and $\text{AR}_{\text{target}}$ for the left-hand sides of equations (3.5a) and (B.4b) respectively.
- 3) Represent $P[D | X = x]$ in (3.5a) and (B.4b) by the robust logistic curve (B.1) (with F_N replaced by \tilde{F}_N).
- 4) Choose initial values for the parameters α and β according to (A.2c), (A.2d) and (A.6).
- 5) Solve numerically the equation system for α and β .

C. Appendix: Minimising the Pearson statistic

Proposition C.1 Assume that in (2.2b) the class probabilities are given as

$$q_i = p g_i + (1 - p) h_i, \quad i = 1, \dots, k, \quad (\text{C.1})$$

where $g_i > 0$ and $h_i > 0$, $i = 1, \dots, k$, are probability distributions and $0 \leq p \leq 1$. Assume that $g_i \neq h_i$ for at least one i . Let furthermore class frequencies n_1, \dots, n_k be given and let $n = \sum_{i=1}^k n_i$. Define the function $f(p)$ by

$$f(p) = \sum_{i=1}^k \frac{(n_i - n(p g_i + (1 - p) h_i))^2}{n(p g_i + (1 - p) h_i)}. \quad (\text{C.2})$$

Then there is a unique $p^* \in [0, 1]$ such that

$$f(p^*) = \min_{0 \leq p \leq 1} f(p). \quad (\text{C.3})$$

We have

- $p^* = 0$ if and only if

$$\sum_{i=1}^k \frac{(h_i - g_i) n_i^2}{(n h_i)^2} \geq 0, \quad (\text{C.4a})$$

- $0 < p^* < 1$ if and only if

$$\sum_{i=1}^k \frac{(h_i - g_i) n_i^2}{(n h_i)^2} < 0 < \sum_{i=1}^k \frac{(h_i - g_i) n_i^2}{(n g_i)^2}, \quad (\text{C.4b})$$

- $p^* = 1$ if and only if

$$\sum_{i=1}^k \frac{(h_i - g_i) n_i^2}{(n g_i)^2} \leq 0. \quad (\text{C.4c})$$

Proof of proposition C.1. The function f is twice continuously differentiable with respect to p :

$$\begin{aligned} f'(p) &= \sum_{i=1}^k \frac{(h_i - g_i) n_i^2}{(n(p g_i + (1-p) h_i))^2}, \\ f''(p) &= \sum_{i=1}^k \frac{(h_i - g_i)^2 n_i^2}{(n(p g_i + (1-p) h_i))^3} > 0. \end{aligned}$$

This last equation implies that f is strictly convex. This implies the assertions because the conditions (C.4a), (C.4b), and (C.4c) are phrased in terms of $f'(0)$ and $f'(1)$. **q.e.d.**

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